Procurement Auctions with Losses

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Abstract

We use a fixed point gradient flow algorithm to compute the equilibria of first-price procurement auctions in the presence of losses and Bayesian priors. We use this efficient algorithm to compare optimal, first-price and VCG auctions. This allows us to numerically estimate the social cost of sub-optimality of the nodal pricing mechanism in wholesale electricity markets. We also derive a closed form expression of the optimal mechanism procurement cost when the types are uniformly distributed.

Keywords: Auctions, Bayesian games, Line losses, Nash Equilibrium, Wholesale electricity market

1 Introduction

Many liberalized electricity markets – such as the US, UK and Spain – involve wholesale trading through an integrated market where frequent market clearing processes take place under the supervision of a central operator, the *Independent System Operator* (ISO). In this context of short-term operations, four notable features arise: (1) demand is inelastic and must be satisfied at each vertex of the grid, (2) the grid imposes significant constraints on the allocations that can be selected, (3) there are losses through the transmission lines, (4) generators have private information about their production cost. Because of those features, generators have a non-negligible degree of *market power* (Borenstein et al, 2000; Escobar and Jofré, 2010).

Indeed, consider one of the most common pricing method, namely *nodal pricing* (also called marginal locational pricing), which consists in the following: ask each generator its cost function, solve the problem of minimizing cost subject to the fulfilment of the demand requirements, then pay, at each node, a price equal to the Lagrange multiplier (shadow price) associated to that demand constraint. With such mechanism prices are local, and hence electricity can be seen as a *geographically differentiated* divisible good, which is a lever of market power.

From the generators' perspective, the choice of the cost function resembles a bidding problem in a Bayesian Bertrand game (or of a *procurement auction*, with the marked difference that the item being sold will partially disappear in the lines): in such a game, strategic behaviors arise. Notably, this is true even if there is complete information among generators but transmission costs are not zero (Escobar and Jofré, 2010).

While auction theory (Krishna, 2009) is a long-standing and active field – pushed nowadays by the e-commerce giants – this is to our knowledge the first work that applies auction theory in a context where *the good can be lost in the allocation*. Also, despite the extensive literature on electricity market equilibria (Hobbs et al, 2000; Hu and Ralph, 2007; Wilson, 2008; Ehrenmann and Neuhoff, 2009) line losses only appear relatively recently in the models (aside from (Borenstein et al, 2000), who note that it can represent 5 to 10% of the flow on the line). The existence of a market equilibrium in a context similar to ours is discussed in (Aussel et al, 2016). A recent paper (Krebs and Schmidt, 2018) shows the uniqueness of market equilibria in the presence of transport costs. A justification of our market model can be found for instance in (Palma-Benhke et al, 2013) and (Wood et al, 2013). Also, because the present study focuses on the losses, we do not add transmission constraints other than the one induced by the line losses (at some point anything that is sent through the line is converted to heat).

This work departs from the standard literature on auction theory by proposing a model with losses. In particular, we do not recover the classical revenue equivalence theorem in symmetric setting. Furthermore, from a mathematical perspective, we observe that the modeling of the line losses regularizes the solution. This allows the design of the algorithm to compute the (Bayes) Nash equilibrium. By contrast, such approach is known to loop for first-price auctions in the classical one item and no line losses setting.

Our contribution is threefold: we adapt Myerson's seminal work (Myerson, 1981) on optimal auction design to line losses and we characterize the mechanism that *minimizes procurement costs* in the presence of incomplete information and transmission losses. We then introduce a fixed point gradient flow algorithm to compute the Nash equilibrium of the game induced by the nodal pricing rule. An example of a run of this algorithm is displayed in Figures 2 and 3. Notably, Figures 2 and 3 reveal the monotonic nature of the corresponding game. The algorithm allows us to benchmark the *Myersonian*

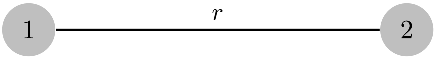

Fig. 1 The binodal market used as a warm-up

rule against the *nodal pricing rule* and Vickrey–Clarke–Groves mechanism (VCG).[1]

Following Borenstein et al (2000) and Escobar and Jofré (2010), we put a focus on the symmetric *binodal* setting, that we use as a warm-up before introducing the general case. Also, we derive a closed form expression of the optimal mechanism procurement cost when the types are uniformly distributed.

2 Auctions with losses

In this section, we introduce the market model and discuss the allocation rule of the central operator when the nodal pricing rule is used.

We remind the reader that a standard, forward single item auction (Krishna, 2009) consists of n buyers that simultaneously bid $(b_v)_{v\in[n]}$ for an item, which is then allocated to the highest bidder in exchange for a payment p that depends on the auction rules. By contrast, here we deal with the procurement auctions of a divisible good, which is how the problem naturally emerges.

Following the approach of Escobar and Jofré (2010), we start with a warm-up on a simple binodal network, where quantities can be explicitly computed, before moving to general networks. In what follows, the network will be represented by an undirected graph (V, E), and for any vertex $v \in V$, $-v$ will be the set of all the other vertex: $-v = V \setminus \{v\}$.

2.1 Warm-up: binodal network

We start with a simple binodal network that has already been studied in Escobar and Jofré (2010). The main ideas sketched in this section will be fully justified in the following section in Theorem 1.

We consider an electric network with two vertices generically denoted by $v \in V$ for $V = \{1, 2\}$, each of which with a fixed demand $d > 0$. At each node, there is a generator with a constant marginal production cost, and it is possible to transmit any amount of electricity $h > 0$ between the vertices through a transmission line. An amount rh^2 shall be lost in the line in this case, for some physical constant $r > 0$. This modeling approach has been proposed previously in the literature to account for the energy that is lost in heat because of the resistivity of the line. More precisely, it corresponds to a quadratic approximation of the loss in the DC approximation (Escobar and Jofré, 2010; Wood et al, 2013). The network looks like the one in Figure 1.

[1]The VCG mechanism is a strategy-proof auction method ensuring participants' best strategy is to bid truthfully, with the highest bidder winning but paying the amount equal to the harm their presence causes to other bidders

The nodal pricing rule consists in the following: the central operator asks each generator its marginal cost of production c_v and then, using the producer's answer b_v as being the true costs, solves the problem of minimizing the total cost $b_1 q_1 + b_2 q_2$ of production — where q_v is the production at vertex v for $v \in \{1,2\}$ — subject to the feasibility constraints (for $v \in \{1,2\}$)

$$q_v - h_{v,-v} + h_{-v,v} \geq \frac{r}{2}[h_{1,2}^2 + h_{2,1}^2] + d. \tag{1}$$

$h_{-v,v}$ corresponds to a positive quantity of energy flowing from $-v$ to v.

The equality constraint corresponds to Kirchhoff's law. What is produced and what goes into the vertex minus what goes out, and the thermal loss should be greater than the local demand. The term $\frac{r}{2}[h_{1,2}^2 + h_{2,1}^2]$ corresponds to the thermal losses (Since at optimum, $h_{1,2}.h_{2,1} = 0$, this is equal to $\frac{r}{2}(h_{1,2}+h_{2,1})^2$), which we assume to be equally covered by the vertices.

If we define

$$G(x,y) = d + \frac{1}{2r}\left(\frac{x-y}{x+y}\right)^2 - \frac{1}{r}\left(\frac{x-y}{x+y}\right), \tag{2}$$

and $\overline{q} = 2\left[\frac{1-\sqrt{1-2dr}}{r}\right]$, then one can show using a standard Lagrangian relaxation argument (Escobar and Jofré, 2010) that the production level $q_v^\star(b_v, b_{-v})$ solution to this problem is

$$q_v^\star(b_v, b_{-v}) = \begin{cases} G(b_v, b_{-v}) & \text{if } G(b_v, b_{-v}) \geq 0 \text{ and } G(b_{-v}, b_i) \geq 0 \\ \overline{q} & \text{if } G(b_v, b_{-v}) \geq 0 \text{ and } G(b_{-v}, b_i) < 0 \\ 0 & \text{else,} \end{cases}$$

and the optimal flow associated to this optimal production plan is

$$\hat{h}^\star_{-v,v}(b) = \begin{cases} \frac{1}{r}\left[\frac{b_{-v}-b_v}{b_{-v}+b_v}\right] & \text{if } b_v \leq b_{-v} \text{ and } G(b_{-v}, b_v) \geq 0 \\ \overline{q} - d & \text{if } b_v \leq b_{-v} \text{ and } G(b_{-v}, b_v) \leq 0 \\ 0 & \text{else.} \end{cases}$$

Then, by definition of the nodal pricing rule, the central operator then asks generator v to produce a quantity $q_v^\star$ and pays him a unit price λ_v, where λ_v is the Lagrange multiplier associated with the feasibility constraint at vertex v:

$$\lambda_v(b_v, b_{-v}) = \begin{cases} b_v & \text{if } G(b_v, b_{-v}) \geq 0 \\ \left[\frac{2-\sqrt{1-2dr}}{\sqrt{1-2dr}}\right] b_{-v} & \text{otherwise.} \end{cases}$$

In our example with one producer per node, such payment rule reduces to pay-as-bid (first-price).

2.2 General Network

We now turn our attention to more generic grids. In what follows, if X and Y are elements of the same Euclidean space, we say that $X \leq Y$ when $x_i \leq y_i$ for all components i. A function f mapping a Euclidean space to another is said to be increasing whenever it preserves the order, that is $X \leq Y$ implies $f(X) \leq f(Y)$. So let (V, E) be an undirected graph, $(d_v)_{v\in V}$ a demand vector and $(r_e)_{e\in E}$ a vector of coefficients for the quadratic losses, that we assume to be identical for both directions of the line e. The optimal allocation problem in this more general setting becomes

$$\min_{h,q\geq 0} \sum_{v\in V} q_v b_v \tag{3}$$

subject to

$$\mathcal{C}_v(q,h) \leq 0 \quad \forall v \in V,$$

with

$$\mathcal{C}_v(q,h) = d_v - \big(q_v + \sum_{u:(u,v)\in E} h_{u,v} - h_{v,u} - \frac{r_{u,v}}{2}(h_{u,v}^2 + h_{v,u}^2)\big).$$

For any procurement cost vector $(b_v)_{v\in V}$ and any vertex $v \in V$, we set, for strictly positive bid vectors:

$$G_v(b_v, b_{-v}) = d_v + \sum_{u:(u,v)\in E} \frac{1}{2r_{u,v}} \left(\frac{b_v - b_u}{b_v + b_u}\right)^2 - \frac{1}{r_{u,v}} \left(\frac{b_v - b_u}{b_v + b_u}\right). \tag{4}$$

It is useful to observe the similarity between Equation (4) and (2). We can now reproduce the result of the warm-up for a general network with Theorem 1.

Theorem 1 *The optimal allocation is equal to*

$${q^\star}_v(b) = G_v(\lambda_v^\star, \lambda_{-v}^\star) \tag{5}$$

where $(\lambda_v^\star)_{v\in V}$ *is the limit of the iterations of*

$$\lambda_v^{(k+1)} = \min(g_v(\lambda_{-v}^{(k)}), b_v), \quad \lambda_v^{(0)} = b_v \tag{6}$$

and where for $\lambda_{-v} \in \mathbb{R}_+^{|V|-1}$, $g_v(\lambda_{-v}) = \min\{\lambda_v \in \mathbb{R}_+ : G_v(\lambda_v, \lambda_{-v}) \leq 0\}$ *is the smallest cost* λ_v *such that* $G_v(\lambda_v, \lambda_{-v}) \leq 0$ *(and* $+\infty$ *if such cost does not exist).*

Proof We proceed in two steps: first we show that the iterations defined in (6) converges, and then, we show the optimality using KKT's sufficient condition.

To see why the iterations defined in (6) converge, we first observe that $G_v(\lambda_v, \lambda_{-v})$ is decreasing in the first variable and increasing in the second variable. Indeed $\partial G_v(\lambda_v, \lambda_{-v})/\partial \lambda_v = -\sum_{u:(u,v)\in E} \frac{4}{r_{u,v}} \frac{\lambda_u^2}{(\lambda_u+\lambda_v)^3} < 0$

and $\partial G_v(\lambda_v, \lambda_{-v})/\partial \lambda_u = \frac{4}{r_{u,v}} \frac{\lambda_u \lambda_v}{(\lambda_u+\lambda_v)^3} > 0$ if u is a neighbors of v, and $\partial G_v(\lambda_v, \lambda_{-v})/\partial \lambda_u = 0$ otherwise.

This implies that g_v is increasing. To see this, take $\lambda_{-v}^1 \leq \lambda_{-v}^2$ and set $\lambda_v^1 = g_v(\lambda_{-v}^1)$ and $g_v(\lambda_{-v}^2) = \lambda_v^2$. Then by definition of g_v, $G_v(\lambda_v^2, \lambda_{-v}^2) \leq 0$, and since G_v is increasing in the second variable and $\lambda_{-v}^1 \leq \lambda_{-v}^2$, $G_v(\lambda_v^2, \lambda_{-v}^1) \leq G_v(\lambda_v^2, \lambda_{-v}^2) \leq 0$. Now with $G_v(\lambda_v^2, \lambda_{-v}^1) \leq 0$ and the definition of g_v, we proved that $\lambda_v^1 \leq \lambda_v^2$. Hence g_v is increasing.

Then, by induction, the iterations $\lambda_v^{(k)}$ are decreasing. Furthermore, these iterations are bounded below by 0 by definition of g_v. Hence, the iterations converge.

Now, we set

$$h_{u,v}^\star = \frac{1}{r_{u,v}} \left(\frac{\lambda_v^\star - \lambda_u^\star}{\lambda_v^\star + \lambda_u^\star} \right)^+ \quad \text{and} \quad q_v^\star = G_v(\lambda_v^\star, \lambda_{-v}^\star).$$

To check the optimality of this putative solution, we introduce $\mathcal{L}(q, h, \lambda, \mu, \gamma)$ the Lagrangian of the problem:

$$\mathcal{L}(q, h, \lambda, \mu, \gamma) = \sum_{v \in V} q_v b_v + \sum_{v \in V} \lambda_v \mathcal{C}_v(q, h) - \sum_{v \in V} q_v \mu_v - \sum_{(v,u) \in E} \gamma_{v,u} h_{v,u}.$$

Then by definition of $\mathcal{L}$, we have

$$\frac{\partial \mathcal{L}(q, h, \lambda, \mu, \gamma)}{\partial q_v} = b_v - \lambda_v - \mu_v \tag{7}$$

and

$$\frac{\partial \mathcal{L}(q, h, \lambda, \mu, \gamma)}{\partial h_{u,v}} = (\lambda_u - \lambda_v) + (\lambda_u + \lambda_v) h_{u,v} r_{u,v} - \gamma_{u,v}. \tag{8}$$

We now make the following observations to show that we can find dual variables $(\mu^\star, \gamma^\star)$ so that the tuple $(q^\star, h^\star, \lambda^\star, \mu^\star, \gamma^\star)$ of primal and dual variables satisfies the KKT conditions.

- If $G_v(\lambda_v^\star, \lambda_{-v}^\star) > 0$ then set $\mu_v^\star = 0$, which implies that $\frac{\partial \mathcal{L}(q,h,\lambda,\mu,\gamma)}{\partial q_v} = 0$. Indeed, suppose $G_v(\lambda_v^\star, \lambda_{-v}^\star) > 0$ then by continuity of G_v, for k greater than some k_0, $G_v(\lambda_v^{(k)}, \lambda_{-v}^{(k)}) > 0$, which implies, by definition of $\lambda^{(k)}$ that $\lambda_v^k = b_v$, hence $b_v = \lambda_v^\star$, which we can plug into relation (7).
- If $G_v(\lambda_v^\star, \lambda_{-v}^\star) = 0$, we can set $\mu_v^\star$ to satisfy $\frac{\partial \mathcal{L}(q^\star,h^\star,\lambda^\star,\mu^\star,\gamma)}{\partial q_v} = 0$ for any γ.
- If $\frac{1}{r_{u,v}} \left(\frac{\lambda_v^\star - \lambda_u^\star}{\lambda_v^\star + \lambda_u^\star} \right)^+ > 0$, then we set $\gamma_{u,v}^\star = 0$, which implies $\frac{\partial \mathcal{L}\left(q^\star, \frac{1}{r_{u,v}} \left(\frac{\lambda_v^\star - \lambda_u^\star}{\lambda_v^\star + \lambda_u^\star} \right)^+, \lambda^\star, \mu^\star, \gamma^\star\right)}{\partial h_{u,v}} = 0$ by relation (8).
- If $\frac{1}{r_{u,v}} \left(\frac{\lambda_v^\star - \lambda_u^\star}{\lambda_v^\star + \lambda_u^\star} \right)^+ = 0$ we can set $\gamma_{u,v}^\star$ so that $\frac{\partial \mathcal{L}\left(q^\star, \frac{1}{r_{u,v}} \left(\frac{\lambda_v^\star - \lambda_u^\star}{\lambda_v^\star + \lambda_u^\star} \right)^+, \lambda^\star, \mu^\star, \gamma^\star\right)}{\partial h_{u,v}} = 0$.

Therefore, by convexity of the problem, and KKT's (sufficient) condition $(q^\star, h^\star)$ solve the optimization problem. □

The economic interpretation is intuitive: if $g_v(\lambda_{-v}^{(k)}) < b_v$ then the producer at vertex v will not be allocated anything (since this would be the case for a linear price of $g_v(\lambda_{-v}^{(k)})$, and the current price is even bigger); if $g_v(\lambda_{-v}^{(k)}) > b_v$

then on the contrary, the producer shall be allocated some production, since the price should be at least $g_v(\lambda_{-v}^{(k)})$ to make its allocation null; last, if $g_v(\lambda_{-v}^{(k)}) = +\infty$, then it means that imports from neighboring vertices cannot cover the demand at vertex v (this can be caused by the line saturated: the quadratic nature of the loss implies that at some point, a marginal amount of electricity that is sent through the line outbound toward v is fully wasted in loss) we are in a situation of monopolistic pricing and inelastic demand, and so without addition to the model, the price at this vertex is likely to be unbounded.

At each step k, the algorithm adapts the prices, until convergence is met.

3 Optimal Mechanism

We now derive Theorem 2, which is an adaptation of (Myerson, 1981) to our context. For the sake of brevity we only provide the main result, since the full proof replicates Myerson's (Myerson, 1981).

We assume the marginal cost c_v of generator $v \in V$ is drawn from a distribution with density f_v (and cumulative F_v) which has full support over $C_v = [c_m, c_M]$. The parameter c_v is known by firm v, but its competitors and the central operator only know the distribution f_v. We set $C = \prod_{v \in V} C_v$ and $f(c) = \prod_{v \in V} f_v(c_v)$.

A *direct mechanism* $M = (q, h, x)$ consists of an *assignment rule* $q : C \longrightarrow (\mathbb{R}_+)^V$ and $h : C \longrightarrow (\mathbb{R}_+)^E$ a *payment rule* $x : C \longrightarrow \mathbb{R}^2$. By the revelation principle[2] (Krishna, 2009) the central operator can restrict to incentive compatible, direct mechanisms, i.e. such that for $c_v, c'_v \in C_v$

$$U_v(c_v, c_v;\ (q, h, x)) \geq U_v(c_v, c'_v;\ (q, h, x)) \tag{9}$$

and

$$U_v(c_v, c_v;\ (q, h, x)) \geq 0 \text{ for all } c_v \in C_v, \tag{10}$$

where U_v is the ex-ante expected utility of a buyer of type c_v when he participates and declares c'_v

$$U_v(c_v, c'_v;\ (q, h, x)) = E_{c_{-v}}[x_v(c'_v, c_{-v}) - c_v q_v(c'_v, c_{-v})].$$

The first set of constraints, (9) are the incentive compatibility constraints: each participant should have no incentive to report anything but its true, *private* production cost. The second set of constraints, (10) are the voluntary participation constraints. They impose that agent should be better off when they participate in the mechanism. Otherwise, one could simply propose the following obviously broken solution, that says that the producer should give their production for free. It is clear that presented with such a proposal, the producer would simply choose to leave the auction.

[2] The revelation principle allows to restrict the search of mechanism to mechanism where the agents are incentivized to tell the truth. This comes from a composition argument: if β is an equilibrium policy for mechanism M, then $M \circ \beta$ is outcome equivalent to M, and incentive compatible.

In the following, we will use the standard simplifying assumption that the function $J_v(c_v) = c_v + \frac{F_v(c_v)}{f_v(c_v)}$, defined on the support of f_v is increasing in c_v. For completeness, we mention that this assumption can be removed using a technical trick that was introduced by Myerson in his seminal paper (Myerson, 1981), but such consideration is out of the scope of this study.[3] Among incentive compatible mechanisms, a mechanism is said *feasible* if it satisfies (9), (10) and

$$q_v(c) + \sum_{u:(u,v)\in E} h_{u,v}(c) - h_{v,u}(c) - \frac{r_{u,v}}{2}(h_{u,v}^2(c) + h_{v,u}^2(c)) \geq d_v \tag{11}$$

for all $c \in C$ and $v \in V$. For an incentive compatible direct revelation mechanism $M = (q, h, x)$, the central operator's expected cost is given by

$$\int\limits_C \sum_{v\in V} x_v(c) f(c) dc. \tag{12}$$

Therefore, the problem of the planner can be written as minimizing (12) subject to (9), (10) and (11):

$$\begin{aligned}
&\min_{(q,h,x)} \int\limits_C \sum_{v\in V} x_v(c) f(c) dc \\
\text{s.t.}\quad & q_v(c) + \sum_{u:(u,v)\in E} h_{u,v}(c) - h_{v,u}(c) - \frac{r_{u,v}}{2}(h_{u,v}^2(c) + h_{v,u}^2(c)) \geq d_v \\
& U_v(c_v, c_v;\ (q,h,x)) \geq 0 \\
& U_v(c_v, c_v;\ (q,h,x)) \geq U_v(c_v, c_v';\ (q,h,x)) \qquad \forall (c, c').
\end{aligned}$$

Myerson's Nobel prize winning result, translated in our context, states that this apparently very complex optimization problem can be solved by leveraging the optimal allocation derived in the previous section (that is, $q^\star$).

Theorem 2 (Myerson's auction) *An optimal mechanism is given by*

$$\begin{aligned}
\hat{q}(c) &= q^\star([J_v(c_v)]_{v\in V}) \\
\hat{h}(c) &= h^\star([J_v(c_v)]_{v\in V}) \\
\hat{x}_v(c) &= c_v \hat{q}_v(c) + \int\limits_{c_v}^{c_M} \hat{q}_v(s, c_{-v}) ds
\end{aligned}$$

The main arguments of the proof of this result, which is an adaptation of Myerson's to our context, are provided in the appendix.

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[3] this technical trick, that relies on a convexification argument, is called the *ironing trick*

4 Nash Equilibrium and algorithm

Next we discuss how we numerically estimate the procurement cost when using the nodal pricing rule.

The nodal pricing rule induces a game. First, the two generators independently draw a marginal cost $c_v \in C_v$ according to f_v, and then play a Bayesian game where the quantity asked from a generator that bids a cost x and confronts a group of generators who bid a cost vector y is given by $q_v(x, y)$, and the unit price paid to him is x. A strategy for the player-generator v is a function $b_v : C_v \longrightarrow \mathbb{R}_+$. Generators maximize their (expected) profit $\pi_v(x, c, b_{-v})$, given by

$$\pi_v(x, c, b_{-v}) = \int (x - c) q_v(x, b(c_{-v})) \mathrm{d}F_{-v}(c_{-v}). \tag{13}$$

Recall that in a Nash equilibrium b^*, player v shall play the profit maximizing bid against b^*_{-v} for the cost he draws, otherwise said, $b^*_v(c) \in \mathrm{argmax}_x \pi_v(x, c, b^*_{-v})$.

Remark 1 By adapting a classical result from forward auctions (see for instance (Krishna, 2009)), to procurement auctions, we see that, for two players, when $r = 0$ and $f_1 = f_2$, the symmetric equilibrium strategy of the game induced by nodal pricing is $b_v(c_v) = \mathbb{E}(c_{-v} \| c_{-v} > c_v)$. The proof mostly relies on the fact that the allocation goes to the lowest value at equilibrium. If we take for instance a uniform distribution on $[0, 1]$, the equilibrium strategy is $c \to 0.5(1 + c)$. For $r > 0$, this observation does not hold anymore, and we could not find a closed form solution.

Computing the Nash equilibrium of a first-price auction is not trivial (Marshall et al, 1994; Wang et al, 2020; Fibich and Gavish, 2012, 2011; Gayle and Richard, 2008) and is a research track in itself. In the symmetric setting, we benefit from a simplification that allows for a closed-form expression (Krishna, 2009). However, such a trick does not apply in our setting because the allocation is not binary.

So we rely on a simple fixed-point gradient flow algorithm (that minimizes for each player the profit given in Equation (13)), described in Algorithm 1 to search for a Nash equilibrium, and display in Figures 2 and 3 the resulting iterations to illustrate its experimental success. A few equilibrium strategies for different values of r are shown in Figure 7. The algorithm receives as parameters a learning rate ϵ, a number of steps N_{iter}, and an initial policy profile b_0.

The algorithm is inspired by the Best Reply iterations for games with strategic complementarities (Topkis, 1998; Vives, 2005). We compensate for the violation of the strategic complementarity assumption with a smoother iteration procedure. Starting from an initial strategy profile that is greater (or smaller) than the equilibrium (using the partial order induced by $\mathbb{R}$), the iterations slowly go down (or up) until convergence is reached, using infinitesimal gradient steps. [4]. The philosophy of the algorithm is summarized in Theorem 3.

Theorem 3 *If the iterations of Algorithm 1 are all increasing (or all decreasing) and bounded, then they converge to a stationary point $b_\star$ of π: $\partial_b \pi_v(b_\star^v(c), c, b_\star^{-v}) = 0$. If, moreover, $b \to \pi_v(b, c, b_\star^{-v})$ is pseudo-concave, then $b_\star$ is a Nash equilibrium.*

Proof The iterations are increasing and bounded, so they pointwisely converge to a point $b_\star^v(c)$. By definition of the algorithm, we should have $\partial_b \pi_v(b_\star^v(c), c, b_\star^{-v}) = 0$. If we suppose in addition pseudo-concavity, $b_\star^v(c)$ maximizes $b \to \pi_v(b, c, b_\star^{-v})$. □

The algorithm was implemented in Python 3. The numerical convergence of the ascending and the descending dynamics to the same strategy profile is an argument for the uniqueness of the equilibrium.

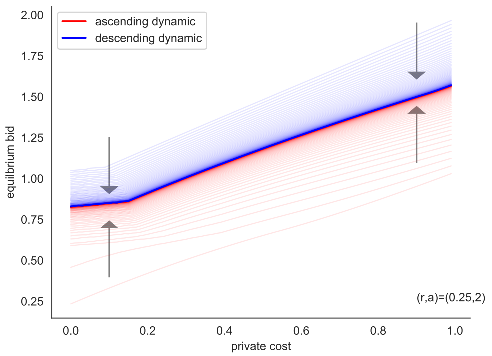

Fig. 2 Convergence of the gradient flow algorithm for $(r, a) = (0.25, 2)$

Algorithm 1 Fixed point gradient flow algorithm

Require: $(\epsilon, b_0, N_{iter})$
  $b \leftarrow b_0$
  for $i \in [N_{iter}]$ **do**
    for $v \in V$ **do**
      $b^v(c) \leftarrow b^v(c) + \epsilon \partial_b \pi_v(b^v(c), c, b^{-v})$
    end for
  end for
  return b

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[4]The connection with games with strategic complementarities is further discussed in this unpublished work https://arxiv.org/abs/2310.02898

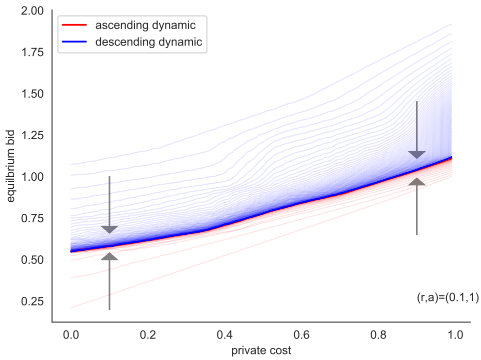

Fig. 3 Convergence of the gradient flow algorithm for $(r, a) = (0.1, 1)$

5 Results

5.1 Simulation

For the gradient flow algorithm, we need an estimation of the gradient of the interim payoff of each producer for each value of the cost:

$$\int q^{\star}{}_v(b_v, b_{-v}(c_{-v}))(b_v - c)\mathrm{d}F_{-v}(c_{-v}). \tag{14}$$

To compute the argument of the multi-integral, we observe that for any b such that for b' in a neighborhood of b, $q^{\star}{}_v(b') = G_v(b'_v, b'_{-v})$, we have $\partial q^{\star}{}_v(b)/\partial b_v = \partial G_v(b_v, b_{-v})/\partial b_v$. When this is not satisfied, we can estimate the gradient using a finite difference. The multiple integrals are estimated with Monte Carlo.

We demonstrate the algorithm on the 6-vertex network displayed in Figure 8. We took $r = 0.05$ and two possible production cost values per node, set at random uniformly between 1 and 2. Convergence was checked by comparing the outcomes of the ascending and descending dynamics of the gradient flow algorithm. The result is displayed on Figure 9.

We now come back to the binodal symmetric setting of Figure 1, which can be considered an extension of a textbook situation, and for which there is an extensive corpus of results in the absence of losses (Vickrey, 1961; Clarke, 1971; Groves, 1973; Krishna, 2009). By the revenue equivalence principle, if $r = 0$ then the optimal pricing, nodal pricing and Vickrey–Clarke–Groves mechanism (VCG) have the same costs. When there are no transmission losses, the central operator procures a single object, which corresponds to the total amount of energy needed in the network. Note that the mechanisms are different, since in the nodal pricing mechanism generators conceal their true cost, while for VCG and Myerson's they reveal their true cost. Still, in expectation, the mechanism yields the same procurement cost.

The equivalence, however, does not hold for $r > 0$. In this case, we have differentiated goods, and the central operator purchases energy from both generators. The central operator solves the dispatcher problem with either the "virtual costs" $c_v + F(c_v)/f(c_v)$ in the optimal mechanism, or the bids $b(c_v)$ in the nodal pricing mechanism, and buys accordingly. Moreover, it pays the established transfers in the former case $x_v(c_v, c_{-v})$ and the asking price $b(c_v)$ in the latter. The mechanisms do not have the same assignment anymore. For two vertices, VCG payment becomes $(\bar{q} - q^\star_{-v}(b_v, b_{-v}))b_{-v}$.

We display in Figures 4 and 5 the procurement and production costs of the nodal, VCG and Myersonian mechanisms as a function of r. We took $f_1 = f_2$ the uniform distribution on $[0, 1]$. We first observed the linear aspect of the optimal procurement cost, which is commented in the next remark. The producer margin seems constant for the optimal procurement mechanism, whereas it spikes for the nodal pricing rule. We also observe that the optimal procurement mechanism is efficient, that is, it has the same production cost as VCG. This is because in a symmetric setting, the optimal procurement mechanism and the VCG mechanism have the same allocation. By contrast, the procurement cost of the Myersonian mechanism is strictly lower than the procurement cost of the VCG mechanism.

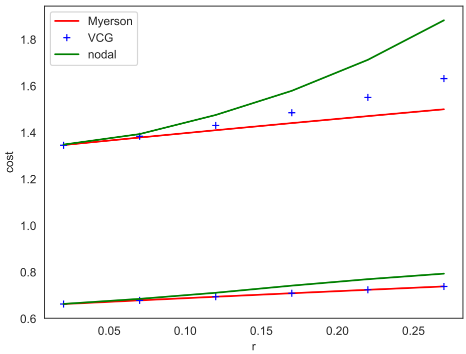

Fig. 4 Production (below) and procurement (above) costs as functions of r for different mechanisms for uniform distribution

5.2 Explicit Procurement Cost for Uniformly Distributed Values

This section is motivated by the surprise we had when computing the optimal procurement cost as a function of r: the result, displayed in Figure 4 looks like an affine function of r. We hence wanted to validate our observation with an explicit computation. The result was quite unexpected: the curve is not a straight line, which is something we verified in Figure 6. When the values are

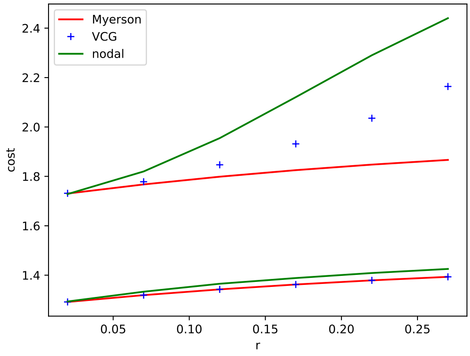

Fig. 5 Production (below) and procurement (above) costs as functions of r for different mechanisms for power law with parameter 2

uniformly distributed on $[0, 1]$, the expected cost paid by the ISO is equal to (see appendix)

$$\frac{4}{3r}\left(\frac{(x_0^2+2x_0)(2rd-1)+x_0^3+2x_0^2+5x_0}{(1+x_0)^2}-2\ln(1+x_0)\right),$$

where $x_0 = 1 - \sqrt{1-2rd}$.

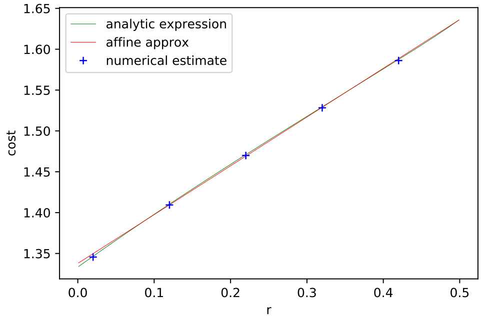

Fig. 6 Nonlinearity of the optimal procurement cost

6 Conclusion

We discussed an extension of auction theory to a graphical setting. This extension was motivated by electricity markets. A unique feature of this model is that some of the good that travels on the network is lost in the process. An interesting side effect of this phenomenon is that the allocation rules are smooth functions of the bids. We identified an algorithm to compute equilibrium that — in the context of our study— revealed "practical" where other approaches failed. We did not, however, provide a sharp characterization of the settings where the monotony of the dynamics will be guaranteed, this shall be an interesting path for further work. Another interesting venue for further investigation could be the design of simple or adaptive mechanisms for this setting.

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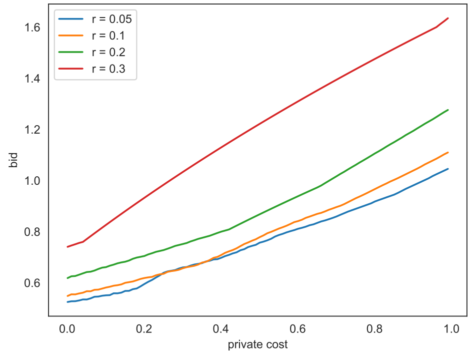

Fig. 7 Some strategies for different values of r. As r comes closer to 0, the numerical scheme requires finer steps.

A proof of the explicit procurement cost formula in Remark 5.2

For $(a, b) \in [0, 1]$, let $\tilde{\mathrm{q}}(a, b)$ be the optimal mechanism allocation (r is omitted to ease the reading).

$$\mathrm{SC}(r) = \int_0^1 \int_0^1 \big(\mathrm{J}(a)\tilde{\mathrm{q}}(a,b) + \mathrm{J}(b)\tilde{\mathrm{q}}(b,a)\big)\mathrm{d}b\mathrm{d}a, \tag{15a}$$

equals, by symmetry, and using the fact that $\mathrm{J}(a) = 2a$ and $\tilde{\mathrm{q}}(a, b) = \mathrm{q}(\mathrm{J}(a), \mathrm{J}(b)) = \mathrm{q}(2a, 2b) = \mathrm{q}(a, b)$

$$2\int_0^1 \int_0^a \big(\mathrm{J}(a)\mathrm{q}(a,b) + \mathrm{J}(b)\mathrm{q}(b,a)\big)\mathrm{d}b\mathrm{d}a = \tag{15b}$$

$$4\int_0^1 \int_0^a \big(a\mathrm{q}(a,b) + b\mathrm{q}(b,a)\big)\mathrm{d}b\mathrm{d}a \tag{15c}$$

which decomposes as the sum of

$$4\int_0^1 \int_{\substack{b<a \\ 0<\mathrm{q}(a,b)}} a\mathrm{q}(a,b) + b\mathrm{q}(b,a)\mathrm{d}b\mathrm{d}a \tag{15d}$$

and

$$4\bar{q}\int_0^1 \int_{\substack{b<a \\ 0=\mathrm{q}(a,b)}} b\mathrm{d}b\mathrm{d}a \tag{15e}$$

Let $X(a,b) = \frac{a-b}{a+b}$, $P(X) = d + \frac{1}{2r}X^2 - \frac{1}{r}X$. Then x_0 is the unique solution of $P(x) = 0$ with $0 \leq x \leq 1$. For $1 > a > b > 0$ $\mathrm{q}(a,b) \geq 0 \iff P(X(a,b)) \geq 0 \iff X(a,b) \leq x_0$. When such a condition is satisfied, $\mathrm{q}(b,a) = P(-X(a,b))$. Therefore (15d) rewrites, using the change of variable $x = (a-b)/(a+b) \iff b = a(1-x)/(1+x)$ and then Fubini's Lemma

$$4\int_0^1 \int_{\substack{0<x<1 \\ x\leq x_0}} \big(aP(x) + a\frac{1-x}{1+x}P(-x)\big)\frac{2a}{(1+x)^2}\mathrm{d}x\mathrm{d}a \tag{15f}$$

$$= 8\int_0^1 a^2\mathrm{d}a \int_0^{x_0} \big(P(x) + \frac{1-x}{1+x}P(-x)\big)\frac{1}{(1+x)^2}\mathrm{d}x$$

$$= \frac{8}{3r}\int_0^{x_0} \frac{2dr - x^2}{(1+x)^3}\mathrm{d}x.$$

Now, since

$$\frac{2dr - X^2}{(1+X)^3} = \frac{2rd-1}{(1+X)^3} + \frac{2}{(1+X)^2} + \frac{-1}{1+X}, \tag{15g}$$

(15d) equals

$$\frac{8}{3r}\left(\frac{2rd-1}{2}\frac{x_0^2 + 2x_0}{(1+x_0)^2} + \frac{2x_0}{1+x_0} - \ln(1+x_0)\right) \tag{15h}$$

Similarly, we get for (15e) that

$$4\bar{q}\int_0^1 \int_{\substack{b<a \\ 0=\mathrm{q}(a,b)}} b\mathrm{d}b\mathrm{d}a \tag{15ia}$$

$$= 4\bar{q}\int_0^1 \int^a b[X(a,b) > x_0]b\mathrm{d}b\mathrm{d}a. \tag{15ib}$$

which is equal, by the change of variable $X(a,b) = x$, to

$$8\bar{q}\int_0^1 a^2\mathrm{d}a \int_{x_0}^1 \frac{1-x}{(1+x)^3}\mathrm{d}x = \tag{15ic}$$

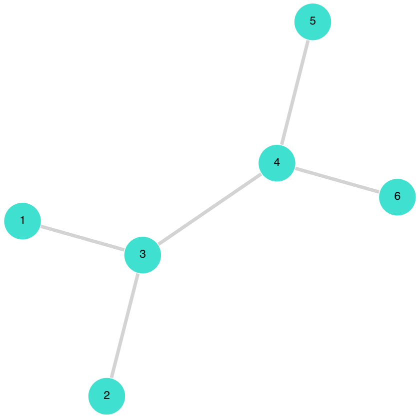

Fig. 8 Network for the numerical experiment of Section 5.

$$\frac{8}{3}\bar{q}\int_{x_0}^{1} \frac{2}{(1+x)^3} - \frac{1}{(1+x)^2}\mathrm{d}x. \tag{15id}$$

We end up with

$$\frac{4x_0}{3r}\left(\frac{1-x_0}{1+x_0}\right)^2. \tag{15ie}$$

Finally (remembering that x_0 is implicitly a function of r)

$$\mathrm{SC}(r) = \frac{8}{3r}\left(\frac{2rd-1}{2}\frac{x_0^2+2x_0}{(1+x_0)^2}+\right.$$

$$\left.\frac{2x_0}{1+x_0} - \ln(1+x_0)\right) + \frac{4x_0}{3r}\left(\frac{1-x_0}{1+x_0}\right)^2.$$

B Steps for the proof of Theorem 2

We now derive Theorem 2, which is an adaptation of Myerson (1981) to our context. For the sake of brevity we only pinpoint the main intermediate results, since the full proof replicates Myerson's.

Lemma 1. *A mechanism (q, h, x) is feasible iff it satisfies (11) and for all $v \in V$*

- $b_v \to \mathbb{E}q_v(b_v, b_{-v})$ *is non-increasing in* b_v,
- $U_v(b_v, b_v;\ (q,h,x)) = U_v(c_M, c_M;\ (q,h,x)) + \int_{b_v}^{c_M} \mathbb{E}q_v(s, b_{-v})\mathrm{d}s$,
- $U_v(c_M, c_M;\ (q,h,x)) \geq 0$.

Proof See proof of Lemma 2 in Myerson (1981). □

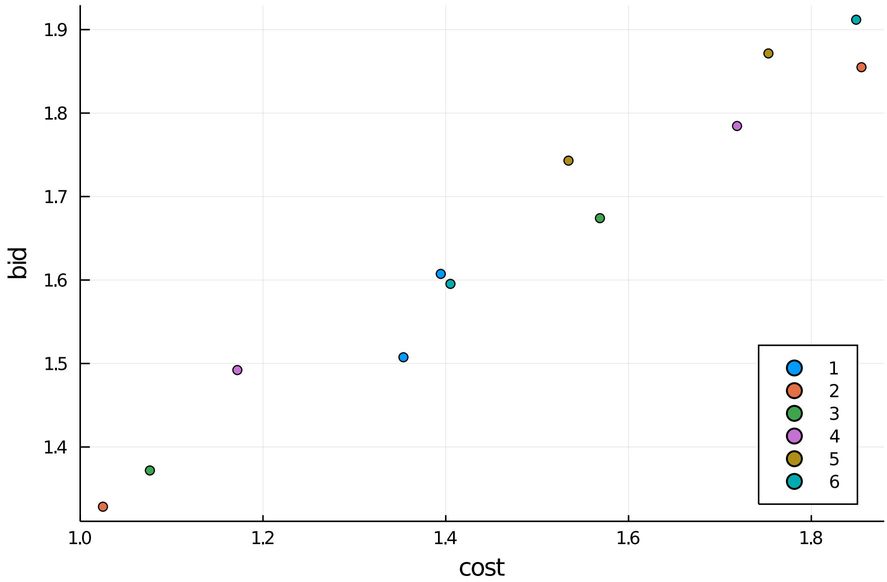

Fig. 9 Result of the equilibrium estimation for the ascending and descending. The color of a point indicates the vertex it corresponds to. The two dynamics coincide.

Next comes a classic reformulation of the expected payment for a feasible mechanism.

Lemma 2. *The expected payment of $v \in V$ in a feasible mechanism is* $\int_C q_v(c)[c_v + \frac{F_v(c_v)}{f_v(c_v)}]f(c)dc$.

Proof For readability, we omit the index v when not needed. If we set $V(s) = U(s, s;\ (q, h, x))$, and $Q : b_v \to \mathbb{E}q_v(b_v, b_{-v})$ then we get, using Lemma 1,

$$\int\limits_C x(c)f(c)dc = \int\limits_C c_v q_v(c_v, c_{-v})f(c)dc + \int\limits_{C_v} V(c_v)dc_v.$$

The second term on the right-hand side can be written (using Fubini's lemma)

$$\begin{aligned}\int\limits_{C_v} V(c_v)dc_v &= \int\limits_{C_v} [V(c_M) + \int\limits_{c_v}^{c_M} Q(s_v)ds_v]f(c_v)dc_v \\ &= V(c_M) + \int\limits_{C_v} Q(s)[\int\limits_{c_m}^{s} f(c)dc]ds \\ &= V(c_M) + \int\limits_{C_v} Q(c)F(c)dc\end{aligned}$$

$$= V(c_M) + \int_{C_v}\int_{C_{-v}} q(c_v, c_{-v})F(c_v)dc_v$$

$$= V(c_M) + \int_{C} q_v(c)\frac{F_v(c_v)}{f_v(c_v)}f(c)dc$$

Replacing this last expression and noticing that in any optimal mechanism $V(\bar{c}_M) = 0$ the result follows.

□

With these two lemmas, we can characterize a solution to the auction design problem.

Lemma 3. *If for a mechanism* $(\hat{q}, \hat{h}, \hat{x})$ *the assignment function* $(\hat{q}, \hat{h})$ *minimizes*

$$\min_{q,h} \int_{C} \sum_{v\in V} q_v(c)[c_v + \frac{F_v(c_v)}{f_v(c_v)}]f(c)dc$$

subject to the constraints that $b_v \to \mathbb{E}q_v(b_v, b_{-v})$ *is non-increasing in* b_v *and the demand constraint* (11)*, and the payment function* $\hat{x}$ *satisfies*

$$\hat{x}_v(c) = \hat{q}_v(c)c_v + \int_{c_v}^{c_M} \hat{q}_v(s, c_{-v})ds$$

then $(\hat{q}, \hat{h}, \hat{x})$ *is an optimal mechanism.*

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