Reverse auctions with transportation and convex costs

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Abstract

We discuss a procurement problem with transportation losses and piecewise linear production costs. We first provide an algorithm based on Knaster-Tarski's fixed point theorem to solve the allocation problem in the quadratic losses case. We then identify a monotony condition on the types distribution under which the Bayesian cost minimizing mechanism takes a simple form.

Keywords: optimal auctions, multidimensional mechanism, allocation algorithm, electricity markets, fixed point

1 Introduction

We discuss the problem of buying a divisible good when the demand and the production are spread on a network. Specifically, the network structure induces losses when the good travels, and potential constraints on the feasibility of the allocation. To be more concrete, the network is made of n nodes (generically denoted by i in the following) connected by edges in $E \subset [n] \times [n]$[1]. On each of those nodes: (a) an inelastic demand d_i is known and (b) there is a producer who has a convex, increasing, piecewise linear production cost.

For simplicity, we assume the existence of $\bar{q}$ such that the changes of slopes of the production cost take place every multiple of $\bar{q}$. Hence, if c_i is the vector

[1] $[k]$ denotes the integers $1 \dots k$

of slopes of producer i (of size m), then her cost for producing q_i is $\sum_{j=1}^{m} q_i^j c_i^j$, where $q_i^j = \min((q_i - (j-1)\bar{q})^+, \bar{q})$.

The model resembles the electricity markets in [1–3], where the network corresponds to the electric grid and where the line losses correspond to the Joule effect. In the context of electricity production, it is very standard to model production cost with convex functions, and in particular piecewise linear convex functions because they interact well with operations research methods. Electricity markets are known to be hard to model because of the counter flows issues [4], but Palma-Benhke et al. [3] provide a condition that shall remove this hurdle, such condition is satisfied, for example, when the network is radial [5] (acyclic). Other models close to our setting were proposed, for example in [6], [7], and [8], with a focus on the existence of a market equilibrium. Distributed markets were also studied in [5, 9], with a focus on efficiency and linear cost for transmissions.

Structure of the Article and Main Contributions

We put under scrutiny the interplay between the line losses and the piecewise linear shape of the production cost. We present our contributions in the two following sections.

First Section 2 introduces a fixed-point algorithm to compute the optimal allocation when losses are quadratic (which in particular corresponds to the setting encountered in [2, 3]). We can interpret the algorithm as if the producers situated at each node of the network were collectively trying to minimize the total cost by communicating their current marginal costs. Such perspective (price decomposition) is a recurring pattern in the optimization of large economic systems.

Second Section 3 presents a condition on the data that reduces the procurement cost minimizing mechanism design problem to a Myerson auction [10], which is quite unusual in a multidimensional setting. Myerson's optimal auctions were originally derived for one item direct auctions with Bayesian priors on the value distributions of each buyer. Because it is cost optimal, we believe the Myersonian mechanism provides a crucial benchmark to discuss procurement mechanisms for divisible goods.
For readability's sake, we defer most of the proofs to the appendix.

2 Study of the procurement problem for quadratic losses

In this section, we state the procurement problem when the losses are quadratic, and derive a fixed point algorithm to solve it. Such a setting corresponds in particular to the one encountered in [2, 3].

2.1 Mathematical program

The buyer needs to buy enough good so that the demand is met at each node. In this section, we assume that when a quantity h of good flows from an undirected edge $(i,i') \in E$, there is a known constant $r_{i,i'}$ such that $r_{i,i'}h^2$ of the good is lost on the way. In the context of electricity markets, this quadratic coefficient corresponds to the Joule effect within the lines [4]. We use $N(i)$ to refer to the nodes adjacent to i in E. With this in mind, the buyer's cost minimizing allocation problem corresponds to the following mathematical program:

Problem 2.1

$$\begin{aligned} \underset{(q,h)}{\text{minimize}} \quad & \sum_{i\in[n]}\sum_{j\in[m]} q_i^j c_i^j \\ \text{subject to} \quad & \sum_{j\in[m]} q_i^j + \sum_{i'\in N(i)} h_{i',i} - h_{i,i'} - \frac{h_{i,i'}^2 + h_{i',i}^2}{2} r_{i,i'} \geq d_i \quad (\lambda_i) \\ & h_{i,i'} \geq 0 \quad (\gamma_{i,i'}), \quad q_i^j \geq 0 \quad (\mu_{i,j}), \quad q_i^j \leq \bar{q} \quad (\nu_{i,j}). \end{aligned} \tag{1}$$

The first set of inequalities correspond to the nodal constraint that demand should be met. We indicate in parentheses the notations we use for the dual variables associated with each constraint. Those variables live in $\mathbb{R}_+$.

2.2 First-order condition

For any node i and $\lambda \in \mathbb{R}^n$ such that $\lambda > 0$, let

$$\mathcal{Q}_i(\lambda_i, \lambda_{-i}) = d_i + \sum_{i'\in N(i)} \frac{\lambda_{i'} - \lambda_i}{r_{i,i'}(\lambda_i + \lambda_{i'})} + \frac{(\lambda_{i'} - \lambda_i)^2}{2r_{i,i'}(\lambda_i + \lambda_{i'})^2}. \tag{2}$$

At the end of this section 2.2, we justify that this function could be interpreted as the production of producer i when the multipliers are λ_i and λ_{-i}.

We proceed with the computation of the dual of Problem 2.1. If a strong duality theorem applies, then we should have

$$\begin{aligned} &\min_{q,h} \max_{\lambda,\gamma,\nu,\mu} \sum_{i,j} q_i^j c_i^j + \sum_i \lambda_i \big(d_i - (\sum_j q_i^j + \sum_{i'\in V(i)} h_{i',i} - h_{i,i'} - \frac{h_{i,i'}^2 + h_{i',i}^2}{2} r_{i,i'})\big) \\ &- \sum_{i,j} \gamma_{i,j} h_{i,j} + \sum_{i,j} \nu_{i,j}(q_i^j - \bar{q}) - \mu_{i,j} q_i^j \\ &= \max_{\lambda,\gamma,\nu\mu} \min_{q,h} \sum_i \lambda_i d_i - \sum_{i,j} \nu_{i,j}\bar{q} + q_i^j(c_i^j + \nu_{i,j} - \lambda_i - \mu_{i,j}) + \\ &\sum_{(i,i')\in E} h_{i,i'}\{\lambda_i - \lambda_{i'} - \gamma_{i,j}\} + h_{i,i'}^2 r_{i,i'} \frac{\lambda_i + \lambda_{i'}}{2}, \end{aligned}$$

so that, by necessary and sufficient first order condition, $h_{i,i'} = \frac{\gamma_{i,i'}+\lambda_{i'}-\lambda_i}{r_{i,i'}(\lambda_{i'}+\lambda_i)}$. Then replacing h by its expression in the dual variables we get something equivalent to

$$\begin{aligned} \underset{(\lambda,\gamma,\mu,\nu)}{\text{maximize}} \quad & \sum_i \big(\lambda_i d_i - \sum_j \nu_{i,k}\bar{q} - \sum_{i'\in V(i)} \frac{(\lambda_i - \lambda_{i'} - \gamma_{i,j})^2}{2r_{i,i'}(\lambda_i + \lambda_{i'})}\big) \\ \text{subject to} \quad & c_i^j + \nu_{i,j} \geq \lambda_i + \mu_{i,j}. \end{aligned} \tag{3}$$

It follows that $\gamma_{i,i'} = (\lambda_i - \lambda_{i'})^+$. The criteria in (3) becomes

$$\underset{(\lambda,\mu,\nu)}{\text{maximize}} \quad \sum_i \big(\lambda_i d_i - \sum_j \nu_{i,j}\bar{q} - \sum_{i'\in V(i)} \frac{(\lambda_i - \lambda_{i'})^2}{4r_{i,i'}(\lambda_i + \lambda_{i'})}\big),$$

and since μ does not play any role in the admissibility of the other variables nor in the objective, the set of constraints in (3) becomes $c_i^j + \nu_{i,j} \geq \lambda_i$. It follows that $\nu_{i,j} = (\lambda_i - c_i^j)^+$. We can now justify that we have strong duality: the operator is continuous, convex-concave and the dual variables are restricted to be in a bounded set. The dual of the allocation problem is therefore written:

$$\underset{\lambda\geq 0}{\text{maximize}} \quad \sum_i \big(\lambda_i d_i - \bar{q}\sum_j (\lambda_i - c_i^j)\mathbb{1}\{\lambda_i \geq c_i^j\} - \sum_{i'\in V(i)} \frac{(\lambda_i - \lambda_{i'})^2}{4r_{i,i'}(\lambda_i + \lambda_{i'})}\big).$$

The problem is now *decomposable*. We maximize, for $i \in I$, the criteria

$$\lambda_i d_i - \bar{q} \sum_{j\in[m]} (\lambda_i - c_i^j)\mathbb{1}\{\lambda_i \geq c_i^j\} - \sum_{i'\in V(i)} \frac{(\lambda_i - \lambda_{i'})^2}{4r_{i,i'}(\lambda_i + \lambda_{i'})}, \tag{4}$$

which is strictly concave for any λ_{-i} (sum of concave and strictly concave functions). We denote by $\Lambda_i(\lambda_{-i})$ its maximizer and set $\Lambda(\lambda_1, ..., \lambda_n) := (\Lambda_1(\lambda_{-1}), ..., \Lambda_n(\lambda_{-n}))$. We get the next lemma by observing that the first order necessary and sufficient condition on Λ_i is $0 \in \mathcal{Q}_i(\Lambda_i, \lambda_{-i}) - K_i(\Lambda_i)$, where

$$K_i(\lambda_i) = \begin{cases} 0 & \text{if } \lambda_i < c_i^1 \\ [j-1, j]\bar{q} & \text{if } \lambda_i = c_i^j \\ j\bar{q} & \text{if } \lambda_i \in]c_i^j, c_i^{j+1}[, j \neq m \\ m\bar{q} & \text{if } \lambda_i \in \lambda_i \in]c_i^m, \bar{c}[. \end{cases}$$

Lemma 1 For any $\lambda^{-i} > 0$, $\Lambda_i(\lambda_{-i})$ is the unique solution of $\mathcal{Q}_i(\Lambda_i, \lambda_{-i}) \in K_i(\Lambda_i)$.

It is insightful to observe that K_i is monotone in the following sense:

$$s < t \implies x \leq y \quad \forall x \in K_i(s), \forall y \in K_i(t)\ . \tag{5}$$

We point out that the primal (and dual) solution uniqueness is a desirable property that is not systematic for the allocation problems of centralized market models. The expression of h with respect to λ together with the supply constraint being binding at optimality justifies the interpretation of $\mathcal{Q}_i$ proposed for its introduction: it is the production of producer i when the multipliers are λ_i and λ_{-i}. We repeatedly use this interpretation in the following.

2.3 Fixed point algorithm

We proceed by showing that the solution of the dual problem is the unique fixed point of the monotone operator Λ. First, we show that Λ is monotone. We then state the Knaster-Tarski's fixed-point Theorem our proof relies on. Follows the two main results of this section: Theorem 7 claims that the solution is the fixed-point of the operator, and Theorem 8 provides an explicit expression for the operator.

Lemma 2 $\lambda_{-i} \to \Lambda_i(\lambda_{-i})$ is non-decreasing.

Proof We proceed ad absurdum: let $(\lambda_{-i}, \lambda'_{-i})$ such that $\lambda_{-i} < \lambda'_{-i}$ and $\Lambda_i(\lambda_{-i}) > \Lambda_i(\lambda'_{-i})$. We observe that we have simultaneously

(a) $\mathcal{Q}_i(\Lambda_i(\lambda_{-i}), \lambda_{-i}) < \mathcal{Q}_i(\Lambda_i(\lambda'_{-i}), \lambda'_{-i})$ because $\mathcal{Q}_i$ is decreasing in the first variable and increasing in the second;

(b) $\mathcal{Q}_i(\Lambda_i(\lambda'_{-i}), \lambda'_{-i}) \leq \mathcal{Q}_i(\Lambda_i(\lambda_{-i}), \lambda_{-i})$ because by Lemma 1, we have $\mathcal{Q}_i(\Lambda_i(\lambda_{-i}), \lambda_{-i}) \in K(\Lambda_i(\lambda_{-i}))$, $\mathcal{Q}_i(\Lambda_i(\lambda'_{-i}),\ \lambda'_{-i}) \in K(\Lambda_i(\lambda'_{-i}))$ and by relation (5), K is increasing;

We conclude by observing that (a) and (b) contradict each other. □

Our main argument relies on the following fixed point theorem [11].

Theorem 3 (Knaster-Tarski fixed point) *Let L be a complete lattice and let f an application from L to L and order preserving. Then the set of fixed points of f in L is a complete lattice.*

In particular, the set of fixed points of an order preserving function cannot be empty. Since Λ is order preserving and any product of compact intervals is a complete lattice when we consider the natural order, there is a fixed point, and the set of fixed points is a lattice.

Lemma 4 λ is optimal for the dual $\Leftrightarrow$ λ is a fixed point of Λ.

Proof If λ is optimal for the dual, then each component i maximizes the criteria (4), thus λ is a fixed point of Λ. Conversely, if λ is a fixed point of Λ, then by definition, each component i maximizes the criteria (4). Hence, since the problem is (strictly) concave, λ is optimal. □

A consequence of the previous lemma and of the dual problem being strictly concave is that

Lemma 5 The set of fixed points of Λ is a singleton.

Lemma 6 For any monotone sequence λ_k converging to a point λ^* in the domain of Λ, $\Lambda(\lambda_k)$ goes to $\Lambda(\lambda^*)$ as k goes to infinity.

The intuition of the proof (that we choose to put in the appendix for clarity) is that we can use the monotony of the sequence and Lemma 1 to characterize the behavior of Λ on the neighborhood. We find that Λ is either constant or characterized by the implicit function theorem.

Theorem 7 *The sequence* $(\Lambda^k(c_1^m...c_n^m))_{k\in\mathbb{N}}$ *converges to the solution of the dual.*

Proof Since $\Lambda(c_1^m...c_n^m) \leq (c_1^m...c_n^m)$, and since Λ is order preserving, the sequence $\Lambda^k(c_1^m...c_n^m) = \lambda^k$ is non-increasing and bounded. Therefore, it converges to a point x. By Berge Maximum theorem [15] for strictly concave criterion Λ is continuous. Therefore $x = \lim_k \lambda^k = \lim_k \Lambda(\lambda^{k-1}) = \Lambda(x)$, *i.e.* x is a fixed point. □

Theorem 8 *For* $\lambda_{-i} > 0$, $\Lambda_i(\lambda_{-i})$ *has the following explicit expression:*

$$\Lambda_i(\lambda_{-i}) = \min\left[\left(A_i^k(\lambda_{-i})\right)_{k=1..m}, \left(B_i^k(\lambda_{-i})\right)_{k=1..m-1}, c_i^m\right] \tag{6}$$

where

$$A_i^k(\lambda_{-i}) = c_i^k \quad \textit{if} \quad \mathcal{Q}_i(c_i^k, \lambda_{-i}) < k\bar{q} \quad \textit{else} \quad +\infty, \tag{7}$$

$$B_i^k(\lambda_{-i}) = \min\left(x \in [c_i^k, c_i^{k+1}];\ \mathcal{Q}_i(x, \lambda_{-i}) = k\bar{q}\right). \tag{8}$$

We can interpret the fixed point algorithm as if some benevolent producers situated at each node of the network were exchanging information. They collectively try to minimize the total cost and, to do so, they communicate their current marginal costs. This marginal cost is the minimum of their local marginal cost and the marginal cost of importation from the adjacent nodes. At each iteration, the producers compute how much they are going to produce based on their current marginal cost. They then update their marginal cost based on the (local) information they just received and transmit this marginal cost to the adjacent nodes.

3 Optimal Mechanism

3.1 Model, problem formulation and separability assumption

We now present a condition that allows the auctioneer to build a Myerson-like auction [10]. We discuss the slightly more general collection of procurement problems:

$$\min_{(h,q)\in\mathcal{A}_1} qc + T(h), \tag{9}$$

where

$$\mathcal{A}_1 = \{(h,q) \in \mathcal{A}_2 \text{ s.t. } g_i(h) + q_i \geq d_i, \ \forall i \in [n]\}$$

corresponds to the demand being met at each node, and

$$\mathcal{A}_2 = \{(h,q) \text{ s.t. } Ah + Bq = b, \ h_{min} \leq h \leq h_{max}, \quad q > 0\}$$

is the network linear constraint. Furthermore, the parameters h_{min} and h_{max} are vectors of size $\mid E \mid$, $T(h)$ is the convex cost associated to the flow plan h. The cost slopes c_i^j are independent random variables, of density f_i^j, cumulative F_i^j, and support $[c_i^{j-}, c_i^{j+}]$, such that the *virtual cost* $J_{i,j} : t \to t + F_i^j(t)/f_i^j(t)$ is increasing. It is possible to remove this regularity condition using the ironing technique introduced in Myerson's seminal paper [10].

We pinpoint that an inequality is used in $\mathcal{A}_1$ instead of the equality encountered in electricity market. Such equality is at the roots of the counter-flow issues [4]. Palma-Benhke et al. [3] provide a condition that shall remove this hurdle, such condition is satisfied, for example, when the network is radial [5].

The auctioneer pays x_i the producer i to produce a quantity q_i. The *payment* x_i depends on the allocation and on the prices proposed by the producers.

A *direct mechanism* is a triple (q, x, h) that maps any cost vector c to a production vector $q(c)$, a payment vector $x(c)$ and a flow $h(c)$. According to the revelation principle [12], we can restrict our search to direct truthful mechanisms. With this notation, the *expected profit* of producer i of type c_i and bid c_i' is $U_i(c_i, c_i') = X_i(c_i') - \sum_{j\in[m]} c_i^j Q_i^j(c_i')$, where the capitalized quantities $Q_i^j(c_i) = \mathbb{E}_{-i} \min((q_i(c_i, c_{-i}) - (j-1)\bar{q})^+, \bar{q})$ and $X_i(c_i) = \mathbb{E}_{-i} x_i(c_i, c_{-i})$ correspond to the average of their non capitalized counterpart over the competition realization.

We can now state the **separability condition**: the *virtual cost* $J_{i,j}(c_i^j)$ is increasing in j for any c_i. The separability condition is different from Myerson's regularity assumption, as the monotony is on j. The virtual cost could be interpreted as the real marginal cost augmented by a marginal information rent. The separability condition imposes the marginal information rent to be such that for any bid, the virtual marginal prices are increasing, *i.e.* the virtual production cost function is convex. The assumption is necessary to show the independence property of the reformulation in Lemmas 17 and 18.

Let us fix a producer i and omit the index i in the notation to help the readability. The **separability condition** asks for the *virtual cost* $j \to J_j(c^j)$ to be increasing in j for any $c = (c^1, c^2, \ldots c^m)$. This implies that for any producer i the sets $\mathcal{W}_j$, where $\mathcal{W}_j = \{J_j(c^j), c^j \in [c^{j-}, c^{j+}]\}$, have zero measure intersections.

Because there is no reason why the producers should willingly report their types, we need to add a constraint on the design to enforce truthfulness. The incentive compatibility (IC) constraints mean that the profit of any producer i of type c_i should be maximal when the producer i bids her true type c_i. In addition, since we want all producers to participate in the market, we need the *participation constraint* (PC). Without this constraint, the auctioneer would optimize as if the producers would accept any deal (even deals where they would make a negative profit). An **optimal mechanism** is a solution of

Problem 3.1

$$\begin{aligned}
&\underset{(q,x,h)}{\text{minimize}} \sum_{i\in I} \mathbb{E}x_i(c) + \mathbb{E}T(h(c)) \\
&\text{subject to:} \quad (q(c), h(c)) \in \mathcal{A}_1 \\
&\forall i, \forall (c_i', c_i): \quad U_i(c_i, c_i) \geq U_i(c_i, c_i') \ (IC) \quad \text{and} \quad U_i(c_i, c_i) \geq 0 \quad (PC).
\end{aligned}$$

We say that this problem is *separable* if it satisfies the separability condition and if the solutions of (9) are unique when the cost vectors of each producer are increasing.

3.2 Partial independence

The next result gives the condition that allows us to adapt Myerson's techniques.

First order condition

We denote by $\mathbb{1}\{\mathcal{A}_1\}$ the support function of $\mathcal{A}_1$ and set $U = \{u = (u_1, \ldots, u_n) \mid u_i \leq 0\}$. Applying Theorem 10.1 from [13], we get that a necessary and sufficient condition for an allocation to be optimal is that

$$0 \in \partial \sum_i \sum_j \min((q_i - (j-1)\bar{q})^+, \bar{q}) c_i^j + \partial T(h) + \mathbb{1}\{\mathcal{A}_1\}(h, q). \tag{10}$$

Now observe that

$$\partial \mathbb{1}\{\mathcal{A}_1\}(h, q) = N_{\mathcal{A}_1}(h, q) \tag{11}$$

$$= \left\{ z - \sum_i y_i \nabla (g_i(h) + q_i)(h, q_i) \mid y \in N_U([g_i(h) + q_i]_i), z \in N_{\mathcal{A}_2}(h, q) \right\}. \tag{12}$$

The last equation requires the qualification constraint (Q) from [14] to be satisfied, so one can use Theorem 4.3 from [14]. Still, note that no matter Q being satisfied, $N_{\mathcal{A}_1}$ does not depend on c.

Theorem 9 *If Problem (3.1) is separable, denote by* $\mathfrak{q}(c)$ *the solutions to* (9). *Then, for any node* $i \in [n]$ *and any* $j \in [m]$,

$$\mathfrak{q}_i^j(s_i^1, \ldots, s_i^{j-1}, c_i^j, s_i^{j+1}, \ldots, s_i^m;\ c_{-i}) = \mathfrak{q}_i^j(t_i^1, \ldots, t_i^{j-1}, c_i^j, t_i^{j+1}, \ldots, t_i^m;\ c_{-i}) \tag{13}$$

Proof Let $c \in \mathcal{C}$. Either $\mathfrak{q}_i^j(c) \in]0, \bar{q}[$ or $\mathfrak{q}_i^j(c) \in \{0, \bar{q}\}$.

First case

If $\mathfrak{q}_i^j(c) \in]0, \bar{q}[$, take $k \neq j$ then c_i^k does not appear in the first order condition (10). By Berge's Maximum Principle [15] the optimal allocation is upper hemicontinuous with respect to the parameter c, by uniqueness of the solution of (9), we get that $c \to \mathfrak{q}_i(c)$ is continuous with respect to c. Thus there is a neighborhood of c_i^k such that q_i^j is still in $]0, \bar{q}[$. In this neighborhood, condition (10) is satisfied for $q_i^j = \mathfrak{q}_i^j(c)$, by uniqueness of the solution, q_i^j is constant with respect to c_i^k on this neighborhood.

Second case

$\mathfrak{q}_i^j(c) \in \{0, \bar{q}\}$. Without loss of generality, let us assume that $\mathfrak{q}_i^j(c) = \bar{q}$. Here, we need to observe that the sub-differential of the criteria with respect to q_i is $[c_i^j, c_i^{j+1}]$, thus the reasoning of the first case can be reproduced whenever $k \neq j+1$. So we only need to deal with the situation where $k = j+1$. Moreover, since q_i is non-increasing in c_i^{j+1}, only an increase of c_i^{j+1} can potentially trigger a change in $\mathfrak{q}_i^j$. Observe that by Berge's Maximum Principle, $\mathfrak{q}_i^j$ is continuous with respect to the parameter of interest c_i^{j+1}. If it happens to take a value different than $\bar{q}$, then this value is also a solution to (10) for the initial parameters, which is in contradiction with the uniqueness of the solution of (10). □

3.3 Main Result

Using Theorem 9, we can now derive the optimal mechanism.

Theorem 10 *Suppose Problem (3.1) is separable. Let (q_i^j, h) be such that $(q_i^j(c_i^j, c_{-i}), h(c))$ minimizes*

$$\sum_{i \in I} \sum_{j \in [m]} q_i^j J_{i,j}(c_i^j) + T(h(c))$$

subject to $(q(c), h(c)) \in \mathcal{A}_1$ and set $q_i(c) = \sum_{j \in [m]} q_i^j(c_i^j, c_{-i})$ and $x_i(c) = \sum_{j \in [m]} q_i^j(c_i^j, c_{-i}) c_i^j + \int_{c_i^j}^{c_i^{j+}} q_i^j(t, c_{-i}) \mathrm{d}t$, then (q, h, x) solves the optimal mechanism design problem.

Hence, an almost direct application of Myerson's auctions is possible for separable problems. The proof of Theorem 10 is provided in the appendix. It is known that multidimensional mechanism design is hard in general [16–18]. We have identified a simple condition that allows us to reduce the problem to the one dimensional setting.

Here is an illustration of why we need the separability condition to hold. We take a modified version of the electricity market model presented in Section 2. Suppose that $r = 0$, $d = 3$, $\bar{q} = 2$ and there is a unique producer of interest who has two production slopes, uniformly distributed in $[0, 1]$ and $[1, 2]$. Hence, the separability condition does not hold. We complete our setting with a competitor who has a fixed, known production cost of $c^- = 1.2$. By definition, $J_1(c_1) = c_1 + c_1/1 = 2c_1$ and $J_2(c_2) = c_2 + (c_2 - 1)/1 = 2c_2 - 1$. Suppose the producer's true type is $(c_1, c_2) = (0.8, 1.5)$, then he will not be allocated anything ($J_1(c_1) > c^-$ and $J_2(c_2) > c^-$) if he bids truthfully, his payoff would hence be 0. By contrast, if the producer announces a type of $(c_1, \hat{c}_2) = (0.8, 1.05)$, he will be allocated 2 units because $J_1(c_1) > c^-$ and

$J_2(\hat{c}_2) < c^-$. The payment would be equal to $2\hat{c}_2 + \int_1^{1.1} 1dt = 2*1.05+0.1 = 2.2$ for a production cost of $c_1*2 = 0.8*2 = 1.6$. Hence, the payoff would be $2.2-1.6 = 0.6 > 0$. Conclusion: the separability condition does not hold, and the auction is not incentive compatible.

A similar issue would occur if instead of defining q as the sum of the q^j, we were to use the relation $q^j = \min((q-(j-1)\bar{q})^+, \bar{q})$ because the producer would be incentivized to increase $\hat{c}_1$ so that $J_1(\hat{c}_1) = J_2(\hat{c}_2)$.

Appendix A Proofs of Section 3

A.1 Notations

For the proof, it is convenient to set $K_i^j(t) = F_i^j(t)/f_i^j(t)$. We denote by $\boldsymbol{C}_i$ the support of types of agent i, and by $\boldsymbol{C}^n$ the product of those supports. We also denote by $V_i(c_i) = U_i(c_i, c_i)$ the expected profit of a producer i if he is of type c_i and bids his true production cost.

The proof relies on the comparison with two intermediary optimization problems:

Problem A.1

$$\underset{(q,x,h)}{\text{minimize}} \sum_{i\in[n]} \mathbb{E}x_i(c)$$

subject to.

$$\forall c \in \boldsymbol{C}^n \quad (q(c), h(c)) \in \mathcal{A}_1 \quad (SD)$$

$$\forall c \in \boldsymbol{C}^n, \forall (i,i') \in E: \quad h_{i,i'}(c) \geq 0$$

$$\forall i \in [n], \forall j \in [m], (c^{-j}, t_1, t_2), (c^1, \dots, t_k, \dots, c^m) \in \boldsymbol{C}_i, : V_i(c^1, .., c^{j-1}, t_1, c^{j+1} .., c^m)$$

$$- V_i(c^1, .., c^{j-1}, t_2, c^{j+1} .., c^m) = \int_{t_1}^{t_2} Q_i^j(c^1, .., c^{j-1}, s, c^{j+1} .., c^m)\mathrm{d}s \quad (H1)$$

$$\forall i \in [n], \forall (c, c') \in \mathcal{C}_i: \quad (c-c').(Q_i(c) - Q_i(c')) \leq 0, \qquad (H2)$$

$$\forall i \in [n], \forall c_i \in \boldsymbol{C}_i: \quad V_i(c_i) \geq 0 \quad (PC),$$

and

Problem A.2

$$\underset{(q,h)}{\text{minimize}} \mathbb{E} \sum_{i\in[n]} \sum_{j\in[m]} q_i^j(c)(c_i^j + K_i^j(c_i^j))$$

subject to

$$\forall c \in \boldsymbol{C}^n \quad (q(c), h(c)) \in \mathcal{A}_1 \quad (SD)$$

$$\forall c \in \boldsymbol{C}^n, \forall (i,i') \in E: \quad h_{i,i'}(c) \geq 0.$$

$$\forall c \in \boldsymbol{C}_i, \forall i \in [n] : x_i(c) = \sum_{j\in[m]} q_i^j(c)c_i^j + \int_{c_i^j}^{c_i^{j+}} q_i^j(c_i^1 \dots c_i^{j-1}, t, c_1^{(j+1)+} \dots c_i^{m+};\ c_{-i})\mathrm{d}t,$$

where the optimization variables (q, x, h) are constrained to be measurable function from $\mathcal{C}^n$ to $\mathbb{R}^n_+$, $\mathbb{R}^n_+$ and $\mathbb{R}^E_+$. The functions Q_i appearing in the constraint (H2) are the vector functions with components Q_i^j, they live in $\mathbb{R}^m$, and the dot in (H2) refers to the scalar product between two vectors of $\mathbb{R}^m$. Problems 3.1 and A.1 are very similar, but (IC) has been replaced by (H1) and (H2) and (PC) is expressed in terms of V instead of U. This replacement is a trick introduced by Myerson in his 1981 paper. We show later on how we can compare Problems 3.1, A.1 and A.2, but note that Problem A.2 is simpler, as the optimization part can be solved pointwise (and x can be deduced from this pointwise optimization). The main result of this paper is that the three problems have the same solution.

A.2 Necessary conditions for Problem 3.1

We derive some necessary conditions for a solution of Problem 3.1. In fact, we only use constraint (IC) to deduce the two next results. The first lemma indicates that any solution of the first problem should be such that Q is monotonous. This is a classic result already introduced in [10]. The novelty here is that in the context of piecewise linear production cost functions, this monotony result is expressed in a vectorial sense.

Lemma 11 (Q monotony) If (q, x, h) is admissible for Problem 3.1, then for all agents $i \in [n]$ and all $(c_i, c_i') \in \boldsymbol{C}_i$

$$(c_i - c_i').(Q_i(c_i) - Q_i(c_i')) \leq 0 \tag{A1}$$

where . is the scalar product in $\mathbb{R}^m$.

Proof We omit the i in the proof, as it plays no role. First, let $(c, c') \in \boldsymbol{C}_i^2$ by the (IC) constraint,

$$U(c, c) \geq U(c, c') \quad \text{and} \quad U(c', c') \geq U(c', c) \tag{A2}$$

i.e.

$$\begin{aligned} X(c) - \sum_{j \in [m]} c^j Q^j(c) &\geq X(c') - \sum_{j \in [m]} c^j Q^j(c') \\ X(c') - \sum_{j \in [m]} c^{j'} Q^j(c') &\geq X(c) - \sum_{j \in [m]} c^{j'} Q^j(c). \end{aligned} \tag{A3}$$

We get the lemma after the summation of the two inequalities and simplification.

□

Lemma 11 indicates that an agent should be producing less on average in his ith working zone if he is bidding a higher marginal cost for this working zone.

Lemma 12 If (q, x, h) is admissible for Problem 3.1 then for any agent (omitting i) for any c, t_1 and t_2

$$\begin{aligned} V(c^1, \ldots, c^{j-1}, t_1, c^{j+1}, \ldots, c^m) =& V(c^1, \ldots, c^{j-1}, t_2, c^{j+1}, \ldots, c^m) \\ & - \int_{t_2}^{t_1} Q^j(c^1, \ldots, c^{j-1}, s, c^{j+1}, \ldots, c^m) \mathrm{d}s \end{aligned} \tag{A4}$$

Proof The inequality $U(c,c) \leq U(c,c')$ implies that $c' \to U(c,c')$ is maximal at c for any $c \in \mathcal{C}_i$. Moreover,

$$t \to U((c^1,..,c^{j-1},t,c^{j+1}..,c^m),c) = X(c) - \sum_{k\in[m]\setminus\{j\}} c^k Q^k(c) - tQ^j(c) \tag{A5}$$

is absolutely continuous, differentiable with respect to t for all c, and its derivative is $-Q^j(c)$. By definition of q^j, $Q^j \leq \bar{q}$. The envelope theorem yields the result. □

A.3 Necessary conditions for Problem A.1

We derive some necessary conditions for a solution of Problem A.1.

Lemma 13 If (q,x,h) is an optimal solution to Problem A.1 then (omitting i) for all $c \in \boldsymbol{C}_i$

$$V(c) = \sum_{j\in[m]} \int_{c^j}^{c^{j+}} Q^j(c^1 \dots c^{j-1}, t, c^{(j+1)+}, \dots, c^{m+})\mathrm{d}t. \tag{A6}$$

Proof According to (H1)

$$\sum_{j\in[m]} \int_{c_j}^{c^{j+}} Q^j(c^1 \dots c^{j-1}, t, c^{(j+1)+}, \dots, c^{m+})\mathrm{d}t =$$
$$\sum_{j\in[m]} V(c^1,..,c^{j-1},c^j,c^{(j+1)+},\dots,c^{m+}) - V(c^1,..,c^{j-1},c^{(j)+},\dots,c^{m+})$$
$$= V(c) - V(c^{1+},\dots,c^{m+}).$$

This is an expression for $V(c)$ as a sum of a positive function of c and a constant $V(c^{1+},\dots,c^{m+})$. It is clear that to optimize the criteria, this constant should be as small as possible. The participation constraint (PC) imposes that $V(c^{1+},\dots,c^{m+}) \geq 0$, therefore $V(c^{1+},\dots,c^{m+}) = 0$. □

A consequence of this is:

Corollary 14 If (q,x,h) is an optimal solution of Problem A.1 then for all $i \in [n]$,

$$V_i(c_i^{1+},\dots,c_i^{m+}) = 0. \tag{A7}$$

Proof See the proof of Lemma 13. □

Another consequence of lemma 13 is

Lemma 15 If (q,x,h) is an optimal solution of Problem A.1, the expected profit of agent i (over his type) is

$$\mathbb{E}V_i(c) = \sum_{j\in[m]} \int_{(c_1..c_n)\in\boldsymbol{C}_i} Q_i^j(c^1,\dots,c^j,c^{(j+1)+},\dots c^{m+})K_i^j(c)f_i(c)\mathrm{d}c. \tag{A8}$$

Proof By Lemma 13 and Fubini's lemma, $\mathbb{E}V_i(c)$ is equal to

$$\mathbb{E}\sum_{j\in[m]}\int_{c^j}^{c^{j+}} Q_i^j(c^1,\dots,c^{j-1},t,c^{(j+1)+},\dots c^{m+})\mathrm{d}t$$

$$=\sum_{j\in[m]}\int_{c^{-j}\in \boldsymbol{C}^{-j}}\int_{c^j=c^{j-}}^{c^{j+}}\int_{t=c^j}^{c^{j+}} Q_i^j(c^1,\dots,c^{j-1},t,c_i^{(j+1)+},\dots c_i^{m+})f_i(c)\mathrm{d}t\mathrm{d}c^j\mathrm{d}c^{-j}.$$

Our task is now to compute the inner term. Applying again Fubini's lemma, this term is equal to

$$\int_{c^j=c^{j-}}^{c^{j+}}\int_{t=c^j}^{c^{j+}} Q_i^j(c^1,\dots,c^{j-1},t,c^{(j+1)+},\dots c^{m+})f_i(c)\mathrm{d}t\mathrm{d}c^j =$$

$$\int_{t=c^{j-}}^{c^{j+}}\int_{c^j=c^{j-}}^{t} Q_i^j(c^1,\dots,c^{j-1},t,c^{(j+1)+},\dots c^{m+})f_i(c)\mathrm{d}c^j\mathrm{d}t =$$

$$\int_{t=c^{j-}}^{c^{j+}} Q_i^j(c^1,\dots,c^{j-1},t,c^{(j+1)+},\dots c^{m+})(\int_{c^j=c^{j-}}^{t} f_i(c)\mathrm{d}c^j)\mathrm{d}t =$$

$$\int_{t=c^{j-}}^{c^{j+}} Q_i^j(c^1,\dots,c^{j-1},t,c^{(j+1)+},\dots c^{m+})(\int_{c^j=c^{j-}}^{t} \frac{f_i(c)}{f_i(c^{-j},t)}\mathrm{d}c^j)f_i(c^{-j},t)\mathrm{d}t =$$

$$\int_{t=c^{j-}}^{c^{j+}} Q_i^j(c^1,\dots,c^{j-1},t,c^{(j+1)+},\dots c^{m+})K_i^j(t)f_i(c^{-j},t)\mathrm{d}t =$$

$$\int_{c^j=c^{j-}}^{c^{j+}} Q_i^j(c^1,\dots,c^{j-1},c^j,c^{(j+1)+},\dots c^{m+})K_i^j(c^j)f_i(c_i)\mathrm{d}c^j\ .$$

We get the lemma by summing all the inner terms. □

Lemma 16 If (H1) is satisfied, then for any $(a,b)\in \boldsymbol{C}_i^2$ (omitting i)

$$X(a)-X(b)=\sum_{j\in[m]}[a^jQ^j(a)-b^jQ^j(b)+\int_{a^j}^{b^j} Q^j(b^1\dots b^{j-1},t,a^{j+1}\dots a^m)\mathrm{d}t]. \quad \text{(A9)}$$

Proof Because of its length the proof is detailed in Appendix B □ □

Lemma 17 If (q,x,h) satisfies (H1) and (H2) and Q_i^j is independent of $c_i^{j'}$ for $j'>j$, then for all $(c,\tilde{c})\in \boldsymbol{C}^2$

$$U(c,c)\ge U(c,\tilde{c}). \quad \text{(A10)}$$

Proof Since (H1) is satisfied, equation (A9) of Lemma 16 applies. We combine this relation with the definition of the expected profit U. We obtain:

$$U(c,c)-U(c,\tilde{c})=\sum_{j\in[m]} c^jQ^j(c)-\tilde{c}^jQ^j(\tilde{c})+$$

$$\int_{c^j}^{\tilde{c}^j} Q^j(\tilde{c}^1,\dots,\tilde{c}^{j-1},t,c^{j+1},\dots c^m)\mathrm{d}t+c^jQ^j(\tilde{c})-c^jQ^j(c)$$

$$= \sum_{j\in[m]} (c^j - \tilde{c}^j)Q^j(\tilde{c}^1, ..., \tilde{c}^{j-1}, \tilde{c}^j)) + \int_{c^j}^{\tilde{c}^j} Q^j(\tilde{c}^1, ..., \tilde{c}^{j-1}, t)\mathrm{d}t$$

$$= \sum_{j\in[m]} \int_{c^j}^{\tilde{c}^j} Q^j(\tilde{c}^1, ..., \tilde{c}^{j-1}, t) - Q^j(\tilde{c}^1, ..., \tilde{c}^{j-1}, \tilde{c}^j)\mathrm{d}t,$$

where we used the independence hypothesis for the second equality. By (H2), which implies the decreasingness of Q^j with respect to c_i^j when all other quantities are fixed, if $c^j < \tilde{c}^j$ then for any $t \in [c^j, \tilde{c}^j]$, $Q^j(t) - Q^j(\tilde{c}^j) \geq 0$. Otherwise, we use the formula $\int_a^b = -\int_b^a$ and that any $t \in [\tilde{c}^j, c^j]$ satisfies $Q^j(t) - Q^j(\tilde{c}^j) \leq 0$. Therefore, $U(c, c) - U(c, \tilde{c})$ is non-negative. □

A.4 Necessary conditions for Problem A.2

We derive some properties for Problem A.2.

Lemma 18 There is an optimal solution (q, x, h) for Problem A.2 such that q_i^j (and Q_i^j) is independent of c_i^k for $k \neq j$.

Proof This is a consequence of the separability assumption and Theorem 9. □

A.5 Proof of Theorem 10

Proof We proceed as follow:

- First note that (q, h, x) is the pointwise solution of Problem A.2 so it is optimal for Problem A.2, moreover, by construction (q, h, x) satisfies (SD) and $h \geq 0$.
- Then note that by Lemma 15, (q, h, x) solves a relaxation of Problem A.1. We need to check that it is admissible for Problem A.1.
- By definition of V (omitting i),

$$V(c_1 \dots a_j \dots c_N) - V(c_1 \dots b_j \dots c_N) =$$
$$\mathbb{E}x(c_1 \dots a_j \dots c_N) - x(c_1 \dots a_j \dots c_m) - [Q^j(a^j)a^j - Q^j(b^j)b^j] =$$
$$\mathbb{E}q_i^j(a^j, c_{-i})a^j + \int_{a^j}^{c_i^{j+}} q_i^j(t, c_{-i})\mathrm{d}t - \mathbb{E}q_i^j(b^j, c_{-i})b^j - \int_{b^j}^{c_i^{j+}} q_i^j(t, c_{-i})\mathrm{d}t$$
$$-[Q^j(a^j)a^j - Q^j(b^j)b^j] = \mathbb{E}\int_{a^j}^{b^j} q_i^j(t, c_{-i})\mathrm{d}t = \int_{a^j}^{b^j} Q_i^j(t)\mathrm{d}t$$

  where we used the definition of x, the definition of Q and Fubini lemma's for the second, third and fourth equalities. Therefore (q, h, x) satisfies (H1).
- By construction, q_i^j is non-increasing in $c_i^j + K_i^j(c_i^j)$ then using the third assumption, q_i^j is non-increasing in c_i^j so for any $(a, b, c_{-i}) \in \boldsymbol{C}^2 \times \boldsymbol{C}^{-i}$, $(a_i^j - b_i^j)(q_i^j(a_i^j, c_{-i}) - q_i^j(b_i^j, c_{-i})) \leq 0$, so by integration with respect to c_{-i},

$(a_i^j - b_i^j)(Q_i^j(a_i^j) - Q_i^j(b_i^j) \leq 0$ and then by summation over j, $(c-c').(Q_i(c) - Q_i(c')) \leq 0$, *i.e.* (H2) is satisfied.

- Since (H1) is satisfied, $V_i(c_i) \geq V_i(c_i^+)$. Moreover, $V_i(c_i^+) = 0$ by construction of x. Therefore, the participation constraint (PC) is satisfied.
- Therefore, (q, h, x) is admissible for Problem A.1. So it solves Problem A.1.
- Since (q, h, x) solves Problem A.1, by Lemma 17 the incentive compatibility constraint (IC) is satisfied. Moreover, by Lemma 13, (PC) is satisfied. Thus, (q, h, x) is admissible for Problem 3.1. We need to check that it is optimal.
- By Lemmas 11 and 12, any optimal solution of Problem 3.1 should be admissible for Problem A.1. Since the criteria are the same, we conclude that (q, h, x) is an optimal solution of Problem 3.1.

□

Appendix B Proof of Lemma 16

Proof By definition

$$\begin{aligned}
X(a^1 \dots a^{k-1}, b, a^{k+1} \dots a^m) - X(a^1 \dots a^{k-1}, c, a^{k+1} \dots a^m) = \\
V(a^1 \dots b \dots a^m) - V(a^1 \dots c \dots a^m) + \\
\sum_{j \neq k} a^j [Q^j(a^1 \dots b \dots a^m) - Q^j(a^1 \dots c \dots a^m)] \\
+ bQ^k(a^1 \dots b \dots a^m) - cQ^k(a^1 \dots c \dots a^m) \\
= \int_b^c Q^k(a^1 \dots s \dots a^m) \mathrm{d}s + \sum_{j \neq k} a^j [Q^j(a^1 \dots b \dots a^m) - Q^j(a^1 \dots c \dots a^m)] \\
+ bQ^k(a^1 \dots b \dots a^m) - cQ^k(a^1 \dots c \dots a^m).
\end{aligned}$$

We use (H1) for the last equality. Then we apply a telescopic formula:

$$\begin{aligned}
X(a) - X(b) = X(a^1 \dots a^m) - X(b^1, a^2 \dots a^m) + \\
X(b^1, a^2 \dots a^m) - X(b^1, b^2 \dots a^m) + \dots \\
+ X(b^1 \dots b^{m^1}, a^m) - X(b^1 \dots b^m) \\
= \sum_{k=1}^{m} \left(\int_{a^k}^{b^k} Q^k(b^1 \dots s \dots a^m) \mathrm{d}s \right) + \\
\sum_{k=1}^{m} \sum_{j<k} b^j [Q^j(b^1 \dots b^{k-1}, a^k, a^{k+1} \dots a^m) - Q^j(b^1 \dots b^{k-1}, b^k, a^{k+1} \dots a^m)] \\
+ \sum_{k=1}^{m} \sum_{j>k} a^j [Q^j(b^1 \dots b^{k-1}, a^k, a^{k+1} \dots a^m) - Q^j(b^1 \dots b^{k-1}, b^k, a^{k+1} \dots a^m)] \\
+ \sum_{k=1}^{m} a^k Q^k(b^1 \dots b^{k-1}, a^k, a^{k+1} \dots a^m) - b^k Q^k(b^1 \dots b^{k-1} \dots b^k, a^{k+1} \dots a^m).
\end{aligned}$$

Reordering the last three terms, we get

$$\sum_{j=1}^{m} \sum_{k>j} b^j [Q^j(b^1 \dots b^{k-1}, a^k, a^{k+1} \dots a^m) - Q^j(b^1 \dots b^{k-1}, b^k, a^{k+1} \dots a^m)]$$

$$
\begin{aligned}
&+\sum_{j=1}^{m}\sum_{k<j} a^j[Q^j(b^1\dots b^{k-1},a^k,a^{k+1}\dots a^m)-Q^j(b^1\dots b^{k-1},b^k,a^{k+1}\dots a^m)]\\
&+\sum_{j=1}^{m} a^jQ^j(b^1\dots b^j-1,a^j,a^{j+1}\dots a^m)-b^jQ^j(b^1\dots b^{j-1}\dots b^j,a^{j+1}\dots a^m)\\
&=\sum_{j=1}^{m}\{b^j\sum_{k>j}[Q^j(b^1\dots b^{k-1},a^k,a^{k+1}\dots a^m)-Q^j(b^1\dots b^{k-1},b^k,a^{k+1}\dots a^m)]\\
&\quad +a^jQ^j(b^1\dots b^{j-1},a^j,a^{j+1}\dots a^m)-b^jQ^j(b^1\dots b^{j-1}\dots b^j,a^{j+1}\dots a^m)+\\
&\quad a^j\sum_{k<j}[Q^j(b^1\dots b^{k-1},a^k,a^{k+1}\dots a^m)-Q^j(b^1\dots b^{k-1},b^k,a^{k+1}\dots a^m)]\}\\
&=\sum_{j}^{m} a^jQ^j(a^1\dots a^m)-b^jQ^j(b^1\dots b^m).
\end{aligned}
$$

We end up with

$$
X(a)-X(b)=\sum_{j=1}^{m}(a^jQ^j(a)-b^jQ^j(b)+\int_{a^j}^{b^j}Q^j(b^1\dots b^{j-1},t,a^{j+1}\dots a^m)\mathrm{d}t). \tag{B11}
$$

$\square$

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